THE CHICAGO MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The fifteenth regular Western Meeting of the American Mathematical Society was held at the University of Chicago on Wednesday and Thursday, December 29 and 30, 1920. This meeting was the forty-sixth regular meeting of the Chicago Section, and was held in affiliation with the Convocation week meetings of the American Association for the Advancement of Science.

The total attendance was more than one hundred, including the following eighty-three members of the Society: Professor R. P. Baker, Professor A. A. Bennett, Professor Henry Blumberg, Professor P. P. Boyd, Professor W. D. Cairns, Professor Florian Cajori, Professor J. A. Caparo, Professor R. D. Carmichael, Professor E. W. Chittenden, Professor C. E .Comstock, Professor H. H. Conwell, Professor D. R. Curtiss, Professor W. W. Denton, Professor L. E. Dickson, Professor Arnold Dresden, Professor L. C. Emmons, Professor H. J. Ettlinger, Professor G. C. Evans, Professor T. M. Focke, Professor G. H. Graves, Professor W. L. Hart, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor T. F. Holgate, Professor Dunham Jackson, Professor O. D. Kellogg, Professor A. M. Kenyon, Professor E. P. Lane, Professor A. O. Leuschner, Mrs. M. I. Logsdon, Professor Gertrude I. McCain, Professor W. D. MacMillan, Professor H. W. March, Professor William Marshall, Professor T. E. Mason, Professor L. C. Mathewson, Professor E. D. Meacham, Professor G. A. Miller, Professor W. L. Miser, Professor U. G. Mitchell, Professor C. N. Moore, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Dr. J. R. Musselman, Professor G. W. Myers, Dr. C. A. Nelson, Mr. H. L. Olson, Professor C. I. Palmer, Professor A. D. Pitcher, Professor L. C. Plant, Professor S. E. Rasor, Professor J. F. Reilly, Professor H. L. Rietz, Professor W. J. Risley, Professor Maria M. Roberts, Professor W. H. Roever, Professor D. A. Rothrock, Dr. A. R. Schweitzer, Professor G. T. Sellew, Professor W. G. Simon, Professor E. B. Skinner, Professor D. E. Smith, Professor E. R. Smith, Professor G. W. Smith, Professor I. W. Smith, Dr. L. L. Steimley, Professor R. B. Stone, Professor E. B.

Stouffer, Professor E. J. Townsend, Dr. B. M. Turner, Professor E. B. Van Vleck, Professor Warren Weaver, Professor W. P. Webber, Mr. F. M. Weida, Professor W. D. A. Westfall, Professor E. J. Wilczynski, Professor C. E. Wilder, Professor D. T. Wilson, Professor R. E. Wilson, Professor C. H. Yeaton, Professor J. W. Young, Professor W. A. Zehring.

The session of Wednesday forenoon was a joint meeting with the Mathematical Association of America and with Sections A and L of the American Association for the Advancement of Science. At this meeting, which was presided over by Professor D. R. Curtiss, chairman of Section A, Professor O. D. Kellogg gave his address entitled *A decade of American mathematics*, as retiring vice-president of Section A of the American Association for the Advancement of Science. His paper was followed by an illustrated paper on *The evolution of algebraic notations*, by Professor Florian Cajori.

The meeting of Wednesday afternoon was held simultaneously with a meeting of the Mathematical Association of America, and was presided over by the chairman of the Chicago Section, Professor R. D. Carmichael. During the Thursday sessions, Professor Carmichael was relieved in the chair by Professor Dunham Jackson, newly-elected vice-president of the Society, and by Professor G. A. Miller.

On Wednesday evening a joint dinner of the Chicago Section, the Mathematical Association of America, the American Astronomical Society, and Sections A and D of the American Association for the Advancement of Science, was held at the Quadrangle Club. At this dinner about one hundred seventy-five persons were present. Professor D. E. Smith, retiring president of the Mathematical Association of America, acted as toastmaster. Short speeches were made by representatives of the different participating organizations. An interesting feature of this dinner consisted of the exhibition by Professor C. I. Palmer of a copy of the 1637 edition of Descartes' Geometry, to which Professor Cajori had alluded in the paper referred to above.

At this meeting the following papers were read:

1. Professor W. P. Webber: *Construction of doubly periodic functions with singular points in the period parallelogram.*

Professor Webber starts with the Weierstrassian p-function and arrives at the function

$$W_2(u) = e^{p(u)} - 1,$$

which is doubly periodic, has zeros at the zeros of $p(u)$ and essential singular points at the poles of $p(u)$. This function is used for constructing more general ones.

It is shown that $W_2(u)$ cannot have an algebraic addition theorem. Among others, the following relations are established: $W_2(u) = [W_2(u) + 1]R(p(u), p'(u))$, where R denotes a rational integral function; $W_2(u) + 1 = e^{F_1(W_2(u),\, W_2'(u))}$, where F_1 is one of the roots of the cubic in $p(u)$, $4\{p^3 - g_2p - g_3\} = \{W_2'(u)/[W_2(u) + 1]\}^2$, g_2, g_3, being the Weierstrassian invariants for $W_2(u)$. Series are derived for $W_2(u)$.

2. Professor H. J. Ettlinger: *Boundary value problems with regular singular points. Second paper.*

In this paper Professor Ettlinger continues his investigation of the oscillatory properties of the solutions of a second order linear boundary value problem, having regular singular points at the end points of the interval.

3. Professor E. W. Chittenden: *Note on the permutability of functions which have the same Schmidt fundamental functions.*

Let $\varphi_i(x)$, $\psi_i(x)$ denote the normalized Schmidt fundamental functions of a kernel $K(x, y) = \Sigma_{i=1}^{\infty} k_i \varphi_i(x) \psi_i(y)$. Professor Chittenden determines the most general function of the form $H(x, y) = \Sigma_{i=1}^{\infty} h_i \varphi_i(x) \psi_i(y)$ which is permutable of the second kind with $K(x, y)$.

4. Professor E. W. Chittenden: *On kernels which have no Fredholm fundamental functions.*

Professor Chittenden studies the classes of kernels which have no Fredholm fundamental functions. For example, if $\varphi_n(x)$, $\psi_n(x)$, $(n = 0, \pm 1, \pm 2, \cdots)$ is a closed normalized biorthogonal system of functions on an interval $(a \leqq x \leqq b)$, the function $K(x, y) = \Sigma_{n=-\infty}^{+\infty} a_n \varphi_n(x) \psi_{n+p}(y)$ (assuming convergence in mean) where the a_n are all different from zero and p is a positive integer, has no Fredholm fundamental functions, and is not orthogonal on the right or left to any function φ such that $\int_a^b \varphi^2 dx > 0$. In case $a_n = 1/n^2$ and the functions φ_n, ψ_n are continuous, the function $K(x, y)$ is continuous.

5. Professor E. W. Chittenden: *Note on convergence in the mean.*

Professor Chittenden presents an example of a sequence of functions which converges in the mean on an interval $(a \leqq x \leqq b)$, but diverges at every point of the interval.

6. Dr. A. R. Schweitzer: *Determination of the spherical transformation in Grassmann's extensive algebra.*

The essence of the geometric application of the quaternions in Hamilton's analysis is to be found in the derivation of Euler's transformation. The attempt of Gibbs to derive these equations by means of Grassmann's Lückenausdrücke appears at a disadvantage when compared with Hamilton's simple deduction based on the transformation qvq^{-1}. Using his article in the MATHEMATISCHE ANNALEN, vol. 69, as a basis, Dr. Schweitzer gives a new derivation of Grassmann's circular transformation and by direct generalization determines the spherical transformation. In this derivation, the simplicity of Hamilton's deduction is completely preserved; it is based on the definitions $X \cdot Q(E_iE_j) = x_1E_1Q(E_iE_j) + x_2E_2Q(E_iE_j) + x_3E_3Q(E_iE_j) + x_4E_4Q(E_iE_j)$, $E_kQ(E_iE_j) = Q(E_kE_i) \cdot E_j$, where $i, j, k = 1, 2, 3, 4$, and where the notation is the same as in the article cited above.

7. Dr. A. R. Schweitzer: *On the relation of iterative compositional equations to Lie's theory of transformation groups.*

The theory of Lie's transformation groups may be interpreted as a theory on the solution of certain functional equations of the iterative compositional type, subject to auxiliary conditions. On this basis, the group property arises through the introduction of a suitable notation. In the MATHEMATISCHE ANNALEN, vol. 18, Lie mentions that his one-dimensional, r-parameter functional equations are a generalization of the equation $\phi\{\phi(x, a), b\} = \phi\{x, \phi(a, b)\}$, which Lie seems wrongly to ascribe to Abel, but which is apparently due to C. J. Hill. Dr. Schweitzer shows that a new definition of transformation groups arises by generalizing the functional equation

$$f\{f(x, a), f(b, a)\} = f(x, b)$$

and that certain of his *quasi-distributive* equations are a degenerate case of Lie's n-dimensional, n-parameter functional equations and that certain of his quasi-transitive functional equations are a degenerate case of the quasi-transitive corre-

lative of Lie's n-parameter, n-dimensional functional equations. On the other hand, Dr. Schweitzer's quasi-transitive equations and their inverse correlatives suggest generalizations of Lie's functional equations and his concept transformation group. The memoirs of Schur in the MATHEMATISCHE ANNALEN treating the analytic and non-analytic solutions of Lie's functional equations readily give direction to the research in the analytic and non-analytic solutions of equations in iterative compositions in general.

8. Professor E. J. Wilczynski: *Isothermally conjugate nets.*

The geometric properties which characterize an isothermally conjugate net were, until recently, entirely unknown. One very elegant characterization was given by the late Dr. G. M. Green. But Green overlooked a case in which his criterion does not distinguish between isothermally conjugate nets and other nets of an entirely different character. In this paper, which has appeared in the AMERICAN JOURNAL OF MATHEMATICS, October, 1920, Professor Wilczynski shows how to complete Green's discussion by introducing an important new concept, namely, that of a *pencil* of conjugate nets, at least in the special case of *isothermally* conjugate nets.

9. Professor E. J. Wilczynski: *Transformation of conjugate nets into conjugate nets.*

In this second paper, Professor Wilczynski shows how the projective invariants and covariants of a conjugate net are affected by transformations which change it into a new conjugate net on the same surface. The notion of a pencil of conjugate nets is here developed in its general form, enabling him to simplify in a notable fashion the results of the preceding paper. But there result, at the same time, two other characteristic properties of isothermally conjugate systems which are entirely different in kind from Green's criterion.

10. Professor R. L. Moore: *Conditions under which one of two given closed linear point sets may be thrown into the other one by a continuous transformation of a plane into itself.*

It is easy to show, by the exhibition of examples, that if S_n is a space of one or more dimensions, there exist in S_n two closed, bounded point sets which are in one-to-one continuous correspondence with each other but neither of which can be

thrown into the other by a continuous one-to-one transformation of S_n into itself. Professor Moore raises the question whether, if S_n is an n-dimensional euclidean space lying in an $n+1$ dimensional euclidean space S_{n+1}, J_1 and J_2 are closed, bounded point sets lying in S_n, and there is a continuous one-to-one correspondence between the points of J_1 and the points of J_2, there exists a continuous one-to-one transformation of S_{n+1} into itself which throws J_1 into J_2. He shows that in the case where $n = 1$ this question can be answered in the affirmative.

11. Professor R. L. Moore: *A closed connected set of points which contains no simple continuous arc.*

Some time ago R. L. Moore and J. R. Kline proposed the following questions: (1) Does there exist a non-degenerate* closed connected set of points which contains no simple continuous arc? (2) Does there exist a non-degenerate connected set of points which contains no arc? In a paper presented to this Society in December, 1919, G. A. Pfeiffer gave an example† showing that the second question may be answered in the affirmative. It is to be noted, however, that not only does the set of points exhibited in this example fail to be closed but it does not even contain a single non-degenerate connected subset that is closed. It, therefore, does not furnish an answer to the first question. In the present paper Professor Moore shows that the first question also may be answered in the affirmative.

12. Professor Florian Cajori: *On the history of symbols for n-factorial.*

Professor Cajori points out the origin of the symbol $\lfloor n$ for n-factorial and traces the spread of this symbol and of $n!$, particularly in the United States. The paper will be published in Isis.

13. Professor L. E. Dickson: *Homogeneous polynomials with a multiplication theorem.*

In his first paper, Professor Dickson investigates all homogeneous polynomials $f(x_1, \cdots, x_n) \equiv f(x)$ of degree d such that $f(x)f(\xi) \equiv f(X)$, where $X_1, \cdots, X_n$ are bilinear functions

* A set of points is said to be non-degenerate if it contains more than one point.

† Cf. this Bulletin, vol. XXVI (1920), p. 246.

of $x_1, \cdots, x_n$ and $\xi_1, \cdots, \xi_n$. Known examples of f are determinants whose ($n = r^2$) elements are independent variables, norms of algebraic numbers, and sums of 2, 4, or 8 squares. It is convenient to introduce the linear algebra in which the product of $x = \Sigma x_i e_i$ by $\xi = \Sigma \xi_i e_i$ is $X = \Sigma X_i e_i$. It is proved that $f(x)$ must divide the dth powers of the determinants $\Delta(x)$ and $\Delta'(x)$ of the general number of the algebra. It is assumed henceforth that $f(x)$ is not expressible in fewer than n variables. By means of a linear transformation on the x's which leaves $f(x)$ unaltered and one on the ξ's leaving $f(\xi)$ unaltered, we secure the simplification that the new composition $x\xi = X$ takes place in a linear algebra having a principal unit e_1 such that $e_1 x = x e_1 = x$ for every x in the algebra. Now every factor of $f(x)$ admits the composition. Next, if $f(x)$ has a covariant of degree δ and index λ which is not identically zero, then f^δ is divisible by Δ^λ and Δ'^λ. Hence, if $\lambda > 0$, f has the same irreducible factors as Δ and Δ', which also admit the composition. For $n < 5$, the hypothesis is satisfied since the Hessian of f is not identically zero (Gordan and Nöther, MATHEMATISCHE ANNALEN, vol. 10, 1876, p. 564). Further, any covariant of f is then a power of f. But an irreducible quartic surface $f = 0$ whose Hessian is cf^2 is a developable surface whose edge of regression is a twisted cubic curve. By means of these theorems it is shown that, if $n < 5$, f is a product of powers of n linear forms or a power of a quaternary quadratic form. For $n = 5$, the former theorems hold, but the case of an irreducible Δ remains undecided. The paper will appear in the PROCEEDINGS OF THE INTERNATIONAL CONGRESS AT STRASBOURG.

14. Professor L. E. Dickson: *Applications of algebraic and hypercomplex numbers to the complete solution in integers of quadratic diophantine equations in several variables.*

In this second paper, Professor Dickson has obtained formulas giving the complete solution in integers of certain equations, as $x_1^2 + x_2^2 + x_3^2 = x_4^2$, $x_1^2 + \cdots + x_5^2 = x_6^2$, when the parameters take only integral values. Transposing one square, we have to express a product as a sum of 2 or 4 squares. Evident solutions are furnished by the theorem on the norm of a product of two complex integers or two quaternions. When the numbers in the resulting formulas are multiplied by the same number ρ and when ρ is allowed to

take all integral values, the products are shown to give all integral solutions. Use is made of the fact that complex integers obey the laws of arithmetic, and likewise quaternions with integral coordinates provided one of the quaternions used has an odd norm. A like theory applies to $x^2 + y^2 + kz^2 = w^2$, when $k = \pm 2, \pm 3, -5, 7, 11, -13$. But if $k = 5$, the numbers $x + y\sqrt{-5}$ do not obey the laws of arithmetic. Corresponding to the two classes of ideals, there are two distinct sets of formulas which together give all integral solutions. This application of ideals will be developed in a later paper. The remaining topics were presented in detail before the International Congress at Strasbourg and will appear in the PROCEEDINGS OF THE CONGRESS.

15. Professor L. E. Dickson: *Arithmetic of quaternions.*

In this third paper, Professor Dickson recalled that A. Hurwitz (GÖTTINGER NACHRICHTEN, 1896, p. 313) proved that the laws of arithmetic hold for integral quaternions, viz. those whose coordinates are either all integers or all halves of odd integers. Since fractions introduce an inconvenience in applications to Diophantine analysis, it is here proposed to define an integral quaternion to be one whose coordinates are all integers. It is called odd if its norm is odd. It is proved that, if at least one of two integral quaternions a and b is odd, they have a right-hand greatest common divisor d which is uniquely determined up to a unit factor ($\pm 1, \pm i, \pm j, \pm k$), and that integral quaternions A and B can be found such that $d = Aa + Bb$. Similarly there is a left-hand greatest common divisor expressible in the form $a\alpha + b\beta$. The further theory proceeds essentially as in Hurwitz's exposition. The paper has been offered to the PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY.

16. Professor L. E. Dickson: *Determination of all general homogeneous polynomials expressible as determinants with linear elements.*

In the fourth paper by Professor Dickson it is proved that every binary form, every ternary form, every quaternary quadratic form, and a sufficiently general quaternary cubic form can be expressed as a determinant whose elements are linear forms, while no further general form has this property. For a plane curve $f = 0$ of order r, we may assume without

loss of generality that f has no repeated factor. Then some line cuts the curve in r distinct points; take it as the side $z = 0$ of a triangle of reference. As the side $y = 0$, take any line not meeting $z = 0$ at one of its r intersections with the curve. Then for $z = 0$, f is a product of r distinct linear functions $X_i = x + \lambda_i y$. It is proved that f can be expressed in one and but one way as a determinant

$$\begin{vmatrix} X_1 + c_{11}z & z & 0 & \cdots & 0 \\ c_{21}z & X_2 + c_{22}z & z & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{r1}z & c_{r2}z & c_{r3}z & \cdots & X_r + c_{rr}z \end{vmatrix},$$

in which the elements above the diagonal are zero except for the elements z just above it. The paper will appear in the TRANSACTIONS. A related paper, dealing with quaternary cubic forms with attention to rationality, has been offered to the AMERICAN JOURNAL OF MATHEMATICS.

17. Professor G. A. Miller: *I-conjugate operators of an abelian group.*

Two operators of any group G are said to be *I-conjugate* if they correspond in at least one of the possible automorphisms of G. As G is supposed to be abelian, it is possible to find a set of generators of $G(s_1, s_2, \cdots, s_\rho)$ such that the group generated by any arbitrary subset of these ρ operators has only the identity in common with the group generated by the rest of them. Any operator of G expressed in terms of these generators is called *I-reduced* if the number of its constituents which are powers of these operators is as small as possible for the set of I-conjugates to which it belongs. Professor Miller then notes that a necessary and sufficient condition that every I-reduced operator of G involve only one constituent is that the quotient of no two invariants of G exceed p when the order of G is p^m, p being a prime number. He also proves that the smallest groups which involve characteristic operators of odd order are two groups of order 63 and that a necessary and sufficient condition that the quotient groups corresponding to the subgroups generated by two I-reduced operators be of the same type, is that the constituents of these operators generate the same subgroups.

18. Professor W. D. Macmillan: *The integrals*

$$\int_0^x e^{x^2/2}dx, \quad \int_0^x \sin\frac{x^2}{2}\,dx, \quad \int_0^x \cos\frac{x^2}{2}\,dx,$$

and associated divergent series.

It is evident that the integral $\int_0^x e^{x^2/2}dx$ cannot be completely tabulated and thus brought into the category of known functions, because it increases beyond all limits and it has no simple properties which enable us to determine its value for a given large value of x. But if we write $\int_0^x e^{x^2/2}dx = e^{x^2/2}\cdot\varphi(x)$, then $\varphi(x)$, for large values of x, is a decreasing function which has the limit zero. It admits the divergent expansion

$$\varphi(x) \equiv \frac{1}{x} + \frac{1!}{x^3} + \frac{1\cdot 3}{x^5} + \frac{1\cdot 3\cdot 5}{x^7} + \cdots,$$

which is very useful for computing the numerical value of φ for large values of x. It is shown by Professor MacMillan that if T_n is the nth term of this series and $\varphi_{n-1}(x)$ is the sum of the first $n-1$ terms of $\varphi(x)$, and if $2n-3 < x^2 < 2n-1$, then T_n is the minimum term and $|\varphi(x) - \varphi_{n-1}(x)| < T_n(x)$; i.e. the error committed in using the divergent series is less than the minimum term provided the series is carried up to, but does not include, the minimum term. Thus, for $x \geqq 6$, the series will give results certainly accurate to 8 decimals, and the order of accuracy increases rapidly with x. A table of values for x up to 6 is given to 7 decimal places. Similar remarks apply to the other integrals.

19. Professor R. P. Baker: *Elementary geometry in n dimensions.*

In this paper, Professor Baker provides a standard method for finding distance and angles between flat spaces in n-dimensional euclidean geometry. Each R_k is supposed given by a matrix $(k+1)(n+1)$ of rectangular coordinates with a column of units. The joint invariants of a pair of such flat spaces are (1) a *mutual moment* involving the distance and all the angles, (2) an *inner product* involving all the angles, and (3) a set of *extensionals* of the general type of mutual moments of figures simply constructed from the data. These furnish symmetric functions of the *slopes*. The algebraic equation for determining the slopes always has real roots.

Since the mutual moment of two lines in space is a polarized form of Plücker's identity, the extensionals are polarized forms of identically vanishing relations (extensionals of Plücker's, etc.) of the determinants of the space. The difficulty of dealing with supernumerary coordinates is surmounted by a wide extension of the theorem of Pythagoras. The *content* of any $(k+1)$ point in R_n is the square root of the sum of the squares of the projections on the axial R_k's. This applies directly to the mutual moment and extensionals and also to the inner product when regression is used to reduce to the case $(k, k, 2k)$. The theory is linear algebra, though regression is used as a device. If progressive methods are desired in connection with the general Pythagorean theorem, we can construct the square of two flat $(k+1)$ points, which is equivalent to the sum of the square of the projections. Models of an R_3 perspective and development of the *square on the triangle* are shown. Finite dissection, however, fails for tetrahedra, Dehn's condition not being satisfied in general.

20. Professor Dunham Jackson: *Note on an ambiguous case of approximation.*

In recent papers, Professor Jackson has discussed the existence and properties of a trigonometric sum $T_{mn}(x)$ of order n at most, determined by the condition that it shall give the best possible approximation to a given continuous periodic function $f(x)$, in the sense of the integral of the mth power of the absolute value of the error. In these discussions it has been assumed that $m \geqq 1$. The purpose of the present note is to inquire what becomes of the problem for values of m less than 1. It is immediately seen that the problem degenerates for $m \leqq 0$, so that there is occasion to consider only values of m between 0 and 1. It is found that there is always at least one determination of $T_{mn}(x)$ which makes the integral a minimum, but that this determination is not generally unique. Nevertheless, it is possible to treat the question of convergence, when m is held fast and n is allowed to become infinite, in substantially the same way as for $m \geqq 1$. In case there are two or more determinations of $T_{mn}(x)$ for a given value of n, it is immaterial for the convergence which is chosen. The treatment applies equally well to the problem of polynomial approximation, and is in part of still wider application.

21. Professor Dunham Jackson: *On the method of least mth powers for a set of simultaneous equations.*

If there is given a set of p simultaneous linear equations in n unknown quantities, $p > n$, the question may be raised of determining values for the unknowns so that the equations shall be approximately solved, in the sense that the sum of the mth powers of the absolute values of the errors is a minimum. For $m = 2$, this is the classical problem of least squares. Professor Jackson treats the general problem by methods analogous to those used in recent papers on the approximate representation of a function of a continuous variable x, the independent variable now being represented by an index ranging from 1 to p. While there is a general correspondence between the two cases, continuous and discrete, both as to methods and as to results, the parallelism in detail does not seem to be so close as to render a separate treatment of the algebraic problem superfluous.

22. Professor Dunham Jackson: *Note on the convergence of weighted trigonometric series.*

In this note, Professor Jackson shows that a method used recently in connection with certain problems of convergence in the theory of trigonometric and polynomial approximations can be applied to a more general class of problems, including some cases considered on the formal side by Gram (CRELLE, vol. 94) in which different weights are assigned to different values of the independent variable.

23. Professor A. J. Kempner: *On polynomials and their residue systems. Second paper.*

In this paper Professor Kempner continues and develops his earlier paper, *On polynomials and their residue systems*, read at the Chicago meeting of the Society in December, 1917. Residue systems of polynomials with respect to a composite numerical modulus are systematically examined.

24. Professor E. R. Smith: *Expansion of the double-frequency function into a series of Hermite's polynomials.*

Dr. Smith gives a development of the general double-frequency function into a series of polynomials which were first investigated by Hermite. The result may be expressed in terms of the successive partial derivatives of the normal

double-frequency function. It is shown that the results are applicable when the regression is non-linear. The generalized form of the equation defining the curve of regression is also discussed.

25. Professor T. E. Mason: *On amicable numbers and their generalizations.*

In this paper Professor Mason presents some new pairs of amicable numbers and some generalized amicable number sets. Dickson defined an amicable k-tuple as k numbers n_1, n_2, $\cdots$, n_k satisfying the equations

$$S(n_1) = S(n_2) = \cdots = S(n_k) = n_1 + n_2 + \cdots + n_k,$$

where $S(n)$ means the sum of all the divisors of n. Amicable k-tuples are given for $k = 3, 4, 5, 6$. Carmichael defined multiply amicable numbers as numbers m and n satisfying the equations $S(m) = S(n) = t(m + n)$. Multiply amicable number pairs are given for $t = 2, 3$, and multiply amicable triples for $t = 2$.

26. Professor Henry Blumberg: *On the complete characterization of the set of points of approximate continuity.*

It is known that the set C of points where a function is continuous is a ΠI_n, i.e. a product of a denumerable sequence of sets I_n, each of which consists exclusively of inner points; conversely, if a set S is a ΠI_n, then a function exists which is continuous at every point of S and discontinuous elsewhere. If we think of continuity as equivalent to the vanishing of the ordinary saltus (or oscillation, or least upper bound minus greatest lower bound), we are led to the notion of *approximate* continuity of different kinds when, in place of the ordinary saltus, we employ the f-saltus, the d-saltus, the z-saltus, etc., which are obtained by regarding as negligible finite sets, denumerable sets, sets of zero measure, etc., respectively. Hence, we are led to seek a characterization of the sets C_f, C_d, C_z, etc., of points of approximate continuity. Professor Blumberg deals with these and related questions. One of the results is that the sets C_f, C_d, C_z are completely characterized as being expressible in the form ΠI_n.

27. Dr. Gladys E. C. Gibbens: *Comparison of different line-geometric representations for functions of a complex variable.*

In the present paper, Dr. Gibbens generalizes the methods given by Professor Wilczynski (TRANSACTIONS, vol. 20) for

constructing a rectilinear congruence by means of a functional relation between two complex variables. She finds that as long as the planes of the two complex variables remain parallel, the projective properties of the class of congruences defined by means of the totality of all analytic functions $w = F(z)$ are independent of the relative position of the origins, of the angle between the real axes of the two complex variables, and of the distance between the planes. The congruence which corresponds to an individual function $w = F(z)$ in any particular representation corresponds not to itself, but to the function $e^{i\theta}w = F(z)$ if the angle between the real axes of the two planes be changed by θ.

If the planes of the two complex variables are not parallel, the congruence defined by the particular functional relation $w = F(z)$ is projectively equivalent to that defined by the function $e^{i\theta_2}w = F(e^{i\theta_1}z)$, where the planes of the new variables $W = e^{i\theta_2}w$; $Z = e^{i\theta_1}z$ are perpendicular to each other, and where the real axes have been rotated through angles θ_1, θ_2, respectively, in order to become parallel to the line of intersection of the planes. The reality of the focal sheets and the developables of the congruence depends upon the particular function under consideration.

If the two complex variables are projected from a common plane upon concentric spheres, and corresponding points joined, the properties of the resulting congruence are again dependent upon the particular functional relation assumed between the two complex variables. In the last two cases, the generalizations are, so far, inadequate from the point of view of a general theory.

28. Professor Dunham Jackson: *On the trigonometric representation of an ill-defined function.*

The ordinary notion of single-valued function of a variable x can be generalized in the following manner: It may be supposed that the value of y corresponding to any given x is indeterminate, and that any value, within bounds, is to be regarded as possible, but that some values are more probable than others, to an extent indicated by a weight which is a function of x and y, essentially positive or zero, That is, such a function of two variables, $\omega(x, y)$, can be regarded as constituting a sort of function of the single variable x, which is blurred or out of focus, and can be made distinct only by a

more or less arbitrary averaging process. Professor Jackson develops the idea thus vaguely expressed, by a study of certain properties of non-negative functions $\omega(x, y)$. It is shown how the method of least squares, or, more generally, of least mth powers, can be used to determine a definite trigonometric sum, of any prescribed order, which may be regarded as representative of the given indeterminate function, supposed periodic with regard to x; and it is proved that, under fairly general hypotheses, the trigonometric sum thus obtained converges uniformly to a suitably defined single-valued average.

29. Professor E. L. Dodd: *An adaptation of Bing's paradox, involving an arbitrary a priori probability.*

Professor Dodd finds that the essential feature of Bing's paradox remains, even after the justly criticized* constant a priori probability is replaced by an arbitrary non-negative continuous probability $w(x)$. If s men of given age all live a moment, the probability that another man of that age will live a moment is $p = \int_0^1 w(x)x^{s+1}dx / \int_0^1 w(x)x^s dx$.

If the s men all live a year of n moments then the probability that the other man also will live a year is p^n, and this approaches zero with increasing n since $p < 1$. It is practically certain therefore, that the other man will die within the year. It should be noted, however, that the use of the same $w(x)$ for a moment of sixty seconds and for a moment of one second can hardly be justified. Thus n, though it may be large, is not subject to indefinite increase.

30. Professor O. D. Kellogg: *A convergence theorem and an application.*

In this note, Professor Kellogg gives a simple proof of a theorem due to Ascoli on the uniform convergence of a subsequence of equi-continuous functions and makes an application to the proof of the existence of a characteristic function for a symmetric kernel.

31. Professors G. D. Birkhoff and O. D. Kellogg: *Invariant points under transformations in function space.*

If a closed convex region of the plane be mapped into itself by a continuous single-valued transformation, there will be

* Arne Fisher, *The Mathematical Theory of Probabilities*, p. 75, quoting Kroman.

an invariant point. Professors Birkhoff and Kellogg propose to extend this theorem to n dimensions and to the space of continuous functions. Certain existence theorems may be formulated in terms of the invariance of a point in a function space under a transformation, and it is proposed to consider such applications of the theorem.

32. Professor G. C. Evans: *Fundamental points of potential theory.*

Professor Evans treats some of the problems of harmonic functions and monogenic functions by means of Lebesgue-Stieltjes integrals and generalized derivatives.

33. Professor W. L. Hart: *Functionals of summable functions.*

Let H represent the class of all functions $u(x)$ on an interval $a \leqq x \leqq b$ which, together with their squares, are summable in the Lebesgue sense on (a, b). In the first part of his paper, Professor Hart considers real-valued functionals $F[u]$ defined for all u in H. It is assumed that if a sequence u_n of H converges in the mean to a function u, then $\lim_{n=\infty} F[u_n] = F[u]$. A representation of $F[u]$ in an infinite series is obtained; a mean-value theorem is obtained for the case that $F[u]$ has a differential; an infinite system of functional equations is solved for an unknown function $u(x, s)$, s being a parameter. In the second part of the paper there are considered functionals $G[u; t]$ which for every u of H yield functions of t that belong to H for $c \leqq t \leqq d$. The concepts continuity and differential are defined. The Fourier coefficients of $G[u; t]$ with respect to a complete orthogonal system $[\varphi_1(t), \varphi_2(t), \cdots]$ are found to be functionals of the type $F[u]$. There are obtained for $G[u; t]$ an infinite series converging in the mean, a mean-value theorem, the solution of implicit functional equations, and a treatment of a pseudo-differential functional equation. In many proofs in the paper, fundamental use is made of theorems and definitions relating to functions of infinitely many variables and to pseudo-derivatives, previously treated by the author.

The papers of Professors Dodd, G. C. Evans, and R. L. Moore, and the paper of Dr. Gibbens, were read by title.

ARNOLD DRESDEN,
Secretary of the Chicago Section.

PLEASANT QUESTIONS AND WONDERFUL EFFECTS.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY, DECEMBER 28, 1920.

BY PROFESSOR FRANK MORLEY.

This is for me a day of reckoning. For us mathematicians most of our days are days of reckoning. But usually, to revive an old phrase, we reckon without our host, without consideration of what we owe and to whom we owe it. I feel that I ought at least to thank the host and to contemplate the question of payment. And at once one sees various hosts.

First and foremost the host is this Society, one of the strongholds of idealism in this country. It was a most happy thought when Columbia men formed the scattered pools of mathematical activity into this important organization. No pool suffered a loss of what may be called potential, many gained enormously. The price must have been paid in the increasing work and sacrifice of the early officers. We took what they provided in the cheerful way of youth. And the youthfulness of the mathematician outlives that of most men, at least if he attends a fair number of meetings of this Society. For he can unload his mind if it is overburdened with a problem too hard for him. And he is sure to go away with some idea, grandiose or neat, obtained in that way of chalk and talk which is the easiest way of getting ideas, to a first approximation.

Or, secondly, the host to whom gratitude should be expressed may be the Engelschaar of great men to whom one owes the science as it stands. But this is obviously too big a field. An interesting point here is the effect of unreasoning veneration on elementary teaching, for example the effect of Euclid on elementary geometry or of Euler on elementary algebra. The adjustment of the properly conservative tradition of teaching to admit important applications is no easy matter. But a good guide in the elementary teaching of any subject is the consideration of its immediate usefulness in neighboring regions, its power of trespass.

Thus arithmetic is rightly taught commercially. Algebra

is on the one hand universal arithmetic, on the other hand it is properly limited by its usefulness in geometry. Geometry should be taught with reference to its many uses. Thus every theorem would have an ultimate root in the ground of intelligent human interest. It would have as its motto "Et documenta damus qua simus origine nati."

The attempt to introduce any branch of mathematics as a pure self-contained logical science is bound to disappoint. I do not, of course, mean this to apply to non-elementary teaching. It is proper to separate the sheep from the goats, but I believe it is a mistake to separate the lambs from the kids.

Or, thirdly, the host to whom one is in private duty bound might be the ideas themselves which caught one's mind and helped to form it. These ideas will be, according to taste, logical or musical or practical. For myself, I confess gratitude to certain simple aids like the chess board or the meridians and circles of latitude on a sphere. With the simple background of a chess board in one's mind, one could survive the onslaught of the rabid algebraists and take a sane interest in determinants and in the elliptic functions of those days.

I have gratitude also for the secondary textbooks in that they gave some small point, some footnotes. You saw where a trail left the high road, where adventure began. This helped you on till you met the books which, being outside the curriculum, seemed pure adventure, as for instance, Whitworth's *Choice and Chance* or Casey's *Sequel to Euclid*, or Clifford's *Kinematics*, or the magnificent works of Salmon. Even the careful, learned, and not over-imaginative Todhunter would give hard and stimulating problems. With an attractive problem one hunted through the theory for a means of solution.

Thus the abstract theory of integration might leave one cold, but the finding of area, or centroid, or length of a curve, was a game. This is, I suppose, what lawyers call the case method, but it is just plain normal mental action.

Soon one saw that most decent theories had their applications, and one would take a chance on theory first. Or one would encounter at college a subject like rigid dynamics, where the problems were hurled at one beyond all reason.

It is possible to combine a reverent study of a reasonable number of classics with a decent development of one's own fragment of mind, to combine a respect for cathedrals with a

liking for making mudpies; but for normal human beings the mudpie is the necessary first step to designing a cathedral or to respecting those who do. And our science needs its fair share of the normal-minded, to spread its ideas, to write its popularizing books. It is becoming too narrowly professional. There are not too many memoirs, but too few readable books.

Or, fourthly, the host may be those who have use for mathematics, the physicist or the statistician. Naturally, they would approve of the less formal introductory teaching which I have advocated. A few clear notions of applied mathematics acquired early will bear a fine superstructure, whereas one who tackles the mathematical treatises on (say) electricity, armed merely with a little analysis and less geometry, has a heart-breaking task if he aims at a real mastery of the mathematical side. So that the infusion of concrete applications, whether it is pedagogically sound as I asserted or not, is indispensable if we are to have a school of applied mathematicians. And this I hold to be a chief desideratum.

In this country research in pure mathematics is solidly rooted and actively efflorescent. Next comes the selection and use of the utilizable. This is the work of the applied mathematician. He and the mathematical physicist are on opposite sides of a counter in a drugstore. The latter seeks from the pharmacopeia what he needs; the former, knowing what the store contains, hands him usually not what he asks for, but a guaranteed article just as good. If more connection can be made between the ordered sense of beauty and power of calculation of the mathematicians and the strong intuitive grasp of our physicists and engineers, it is certain that wonderful effects will follow.

Fifth among the hosts should be the one subject which a man, now supposed to be grown, selects as his intellectual home. I venture to speak briefly for geometry on the ground of having worked diligently at relatively simple geometric questions.

That it should be treated logically is all right; but do not vivisect it merely for the sake of logic. By all means let it have foundations; but few will enter thereby. For it is properly the great exemplar of applied mathematics. And thanks to many workers, of whom Einstein is the latest and the most conspicuous, its future will be glorious.

The covariants and contravariants in which the projective

and inversive geometers have delighted, but which they have signally failed to popularize, are now to be almost a commonplace in the wider field of differential geometry. The physicists who would not look at the projective gnats will swallow the differential camels. And there is no doubt more nutriment in a camel. There will be a lively market, and it should be met by some recasting where pedagogic reasons already exist. The mapping of spaces should be led up to by the simplest cases of mapping. Once again, what pedagogy sighed for physical science demands, this time in the field of elementary geometry itself.

If the science should be taught in its early stages not as a jumble of special applications, but always with an honest consideration of its legitimate contexts, then would it still be true of the far wider mathematics of today, that, to quote old Isaac Barrow again, "The Mathematics is the unshaken Foundation of Science and the Plentiful Fountain of Advantage in Human Affairs."

JOHNS HOPKINS UNIVERSITY.

FALLACIES AND MISCONCEPTIONS IN DIOPHANTINE ANALYSIS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society March 26, 1921.)

§ 1. *Introduction.* Numerous writers have claimed to find all integral solutions of various homogeneous equations when they have actually found merely the rational solutions, expressed by formulas involving rational parameters. They have really left untouched the more difficult problem of finding all the integral solutions exclusively. The fallacies exposed in § 2 and § 3 are merely particular instances of the wide-spread misconception of the problem of solving a homogeneous equation in integers. It is therefore not safe, without reexamination, to place confidence in any claim that a homogeneous equation has been completely solved in integers.

In the next number of this BULLETIN, I shall show how the

theory of ideals can be applied to find all the solutions in integers of the homogeneous equation $x^2 + ay^2 + bz^2 = w^2$.

§ 2. *A Fallacy concerning Pairs of Equations.* It has been regarded as self-evident by all writers,* who have mentioned the topic, that the problem of solving a non-homogeneous equation in rational numbers is equivalent to the problem of solving the corresponding homogeneous equation in integers. Let us examine this question for the particular homogeneous equation

$$x^2 + 5y^2 = zw \tag{1}$$

and the corresponding non-homogeneous equation

$$X^2 + 5Y^2 = Z. \tag{2}$$

The problem of solving the latter in rational numbers is trivial. But the problem of solving (1) in integers involves the finding of all divisors of all numbers that can be represented by $x^2 + 5y^2$, which is one of the serious questions in the theory of quadratic forms. This problem will be treated in the next number of the BULLETIN by the theory of ideals; the conclusion is quoted at the end of § 4 below.

It is clear that there must be some fallacy in the customary argument that two such problems are equivalent.† This argument is the following simple one. If x, y, z, w $(w \neq 0)$ are integers satisfying (1) and if we write

$$\frac{x}{w} = X, \qquad \frac{y}{w} = Y, \qquad \frac{z}{w} = Z, \tag{3}$$

we obtain rational numbers satisfying (2). Conversely, if X, Y, Z are rational numbers satisfying (2), we may express them as fractions (3) with a common denominator and obtain integers x, y, z, w satisfying (1).

Here there is nothing wrong with the algebraic work, nor with the facts deduced. The fallacy lies in the failure to perceive that these facts do not warrant the conclusion that, in the converse case, we have shown how to find all integral solutions. That goal requires that we find all integers w such that the products wX, wY, wZ are integers, viz., x, y, and z. All such integers w are evidently multiples of the minimum

* Including Gauss, *Disquisitiones Arithmeticae*, § 300.

† Namely, that any solution of one equation corresponds to solutions of the other equation under the transformation (3).

positive integer w. To find the minimum w, we need the least common denominator l of the fractions X, Y, and Z. Let d denote the least common denominator of the fractions X and Y, so that $X = \xi/d$, $Y = \eta/d$, where ξ, η, and d are integers without a common factor > 1. Then we have

$$Z = \frac{\xi^2 + 5\eta^2}{d^2}.$$

Before we can find l, we must find the irreducible fraction which equals Z. But this requires the knowledge of all the divisors of all numbers that can be represented by $\xi^2 + 5\eta^2$. Hence we have made no real advance over our initial problem (1) by utilizing our knowledge of the complete solution in rational numbers of the corresponding non-homogeneous equation (2).

§ 3. *The Fallacy when both Equations are Homogeneous.* There is a wide-spread belief that the problem of finding all rational solutions of a homogeneous equation is equivalent to that of finding all its integral solutions. The argument was recently restated by a specialist as follows: (i) the set of all rational solutions contains the set of all integral solutions, and (ii) from the set of all integral solutions it is obvious that the set of all rational solutions is obtained by dividing the numbers in each solution by an arbitrary positive integer.

But remark (i) does not serve the purpose intended, since it leaves unanswered the vital question of how to select the infinitude of integral solutions from the rational solutions. The futility of the argument is emphasized by replacing (i) by the equally trivial remark that all integral solutions occur among the real (or complex) solutions.

In order to bring out clearly the distinction between the two problems, consider the special equation (1). Its rational solutions are obviously all included in the following two types: $x = y = z = 0$, with w any rational number; and x, y, z any rational numbers such that $z \neq 0$, with $w = (x^2 + 5y^2)/z$. We have therefore solved by inspection our first problem of finding all the rational solutions.

Does this information alone serve, as claimed, to yield the complete solution of our second problem of finding all the integral solutions of equation (1)? If so, we should be able, without further theory, to pick out the integral solutions from

the preceding rational solutions. This is easily done for the first type of rational solutions; we have only to restrict w to integral values. For the second type, we must not only restrict x, y, and z to integral values, but we must also examine the condition that $x^2 + 5y^2$ shall be divisible by z. Expressed otherwise, we require a process, valid for arbitrary integers x and y, of finding all divisors z of $x^2 + 5y^2$ (the quotients giving the corresponding values of w). Since we have merely returned to a restatement of our second problem of finding all the integral solutions of (1), we have made no advance whatever on that problem by considering the first problem of finding the rational solutions.

§ 4. *A common Misconception concerning Integral Solutions of a Homogeneous Equation.* To have a concrete case in point, let us express the rational solutions of equation (1) in the customary homogeneous form, which has the advantage of combining into a single formula the two preceding types of solutions. For $z \neq 0$, express x, y, and z as fractions with the positive least common denominator l, and let n be the greatest common divisor of the numerators. Then

$$(4) \quad x = \frac{na}{l}, \quad y = \frac{nb}{l}, \quad z = \frac{nc}{l}, \quad w = \frac{n(a^2 + 5b^2)}{cl},$$

where a, b, and c are integers without a common factor > 1, while n and l are integers without a common factor > 1, and $cl \neq 0$. Write ρ for $n/(cl)$. Then

$$(5) \quad x = \rho ac, \quad y = \rho bc, \quad z = \rho c^2, \quad w = \rho(a^2 + 5b^2).$$

The solutions with $z = 0$ have $x = y = 0$ and are of the form (5) with $c = 0$. Hence all rational solutions of (1) are given by (5), in which a, b, and c are integers without a common factor, while ρ is rational.

Some writers are in the habit of suppressing the *proportionality factor* ρ and claiming without further examination that the resulting values give the general solution in integers. Essentially the same error vitiates the claim of Desboves* that he obtains the complete solution in integers of the general homogeneous quadratic equation in n unknowns when one solution $x, y, \cdots$ is given. Since he regarded $mx, my, \cdots$ as the same solution as $x, y, \cdots$, where m is rational, it is clear

* Nouvelles Annales, (3), vol. 3 (1884), pp. 225–39.

that he found at most formulas for the rational* solutions. Thus he deliberately prevented himself from even attacking the far more difficult problem of finding the integral solutions, though he claimed to find them.

For most homogeneous equations the true state of affairs is analogous to what we shall show to be the case for our special equation (1). Unfortunately we do not obtain all integral solutions if we restrict ρ to integral values in (5), but we must employ values whose denominators increase without limit.

By a certain simplification we shall place in its most favorable light the question of describing all sets of numbers $a, b, c,$ and ρ (with $a, b,$ and c integers without a common factor, and ρ rational) for which the solution (5) is integral, and we shall show that there remains an essential difficulty in the determination of these sets. First, if $c = 0$, then $x = y = z = 0$, and w may be identified with any assigned integer k by taking $\rho = k$, $a = 1$, $b = 0$, for example. Next, let $c \neq 0$. Returning from (5) to the equivalent form (4), we see that x, y, z, w are integers if and only if $l = +1$ and $n(a^2 + 5b^2)$ is divisible by c, whence $\rho = n/c$. Eliminating n, we see† that the conditions on $a, b, c,$ and ρ are that ρc and $\rho(a^2 + 5b^2)$ be integers.‡ Hence the infinitude of sets of numbers $a, b, c,$ and ρ for which formulas (5) give integers, and hence give all integral solutions of (1), may be *described* as follows: (i) the sets $a = 1$, $b = c = 0$, with ρ integral; (ii) the sets for which $a, b,$ and c ($c \neq 0$) range over all triples of integers without a common factor, while for each triple ρ ranges over all the irreducible fractions whose denominators are common divisors of c and $a^2 + 5b^2$. But we have not shown how to *determine* the sets $a, b, c,$ and ρ just described. Their determination requires the finding of all divisors of all numbers represented by the quadratic form $a^2 + 5b^2$. Our simplified description of the integral solutions on the basis for the formulas (5) for the

* But these can be found at once by considering all the lines through the given rational point.

† Also direct from (5) by using the theorem that, if a, b, c have the greatest common divisor 1, integers A, B, C may be found such that $aA + bB + cC = 1$. Multiply by ρc and apply (5). Thus $xA + yB + zC = \rho c =$ integer.

‡ It is now easily proved that the denominators of the ρ's are unlimited. As is known, there is an infinitude of primes p of the form $\alpha^2 + 5\beta^2$. To obtain the solution $x = \alpha, y = \beta, z = p, w = 1$ by (5), we must take the integral factors ρc and c of z to be ± 1 and $\pm p$, whence $\rho = 1/p$, since the choice $\rho c = \pm p$, $c = \pm 1$ would give $\rho = p$, $w > 1$.

rational solutions is therefore no essential improvement upon the description which the proposed equation itself may be said to give.

In accord with the theory to be explained in the next number of this BULLETIN, the successful determination of the integral solutions is made on the basis of a study, not of formulas (5) for the rational solutions, but of the new formulas

$$\begin{aligned} x &= \rho(ac - 5bd), \qquad z = \rho(c^2 + 5d^2), \\ y &= \rho(ad + bc), \qquad w = \rho(a^2 + 5b^2), \end{aligned} \tag{6}$$

which reduce to (5) when $d = 0$ and hence give all the rational solutions. What really happens is well explained in the language of medicine: the injection of the additional integral parameter* d into our solution (5) counteracts the irritation caused by the rational ρ's with their infinitude of denominators. To prevent confusion in a comparison with (6), rewrite (6) in new letters:

$$\begin{aligned} x &= \sigma(AC - 5BD), \qquad z = \sigma(C^2 + 5D^2), \\ y &= \sigma(AD + BC), \qquad w = \sigma(A^2 + 5B^2). \end{aligned} \tag{7}$$

We now attempt to describe all sets of numbers A, B, C, D, and σ (with A, B, C, and D integers without a common factor, and σ rational) for which formulas (7) give integers and hence give all integral solutions of (1). When σ is integral, there is no additional restriction on the integers A, B, C, and D. When σ is an irreducible fraction with the denominator 2, the numbers (7) are all integers if and only if $C \equiv D$, $A \equiv B \pmod 2$. Hence we write

$$D = q,\ C = 2l + q,\ B = r,\ \ A = 2n - r,\ \sigma = \tfrac{1}{2}\rho,$$

and we obtain

$$\begin{aligned} x &= \rho(2ln - lr + nq - 3qr), \qquad y = \rho(lr + nq), \\ z &= \rho(2l^2 + 2lq + 3q^2), \qquad w = \rho(2n^2 - 2nr + 3r^2), \end{aligned} \tag{8}$$

* It is rationally redundant. Any given solution (7), which is (6) written in new letters can be expressed in the form (5). If $C = D = 0$, whence $x = y = z = 0$, take $c = 0$ and identify the two w's, which can be done in infinitely many ways. If C and D are not both zero, take

$$a = (AC - 5BD)/t, \quad b = (AD + BC)/t, \quad c = (C^2 + 5D^2)/t, \quad \rho = t\sigma/c,$$

where t is the greatest common divisor of the three numbers whose division by t is indicated. Our former conditions that ρc and $\rho(a^2 + 5b^2)$ be integers now require that σt and $\sigma(A^2 + 5B^2)$ be integers.

where l, q, n, and r are integers without a common factor, and ρ is an integer.* Next, if σ is an irreducible fraction $\rho/5$ with the denominator 5, the numbers (7) are all integers if and only if C and A are divisible by 5. Writing $A = 5b$, $C = -5d$, $B = -a$, $D = c$, we obtain (6). A more typical case is that in which σ is an irreducible fraction of the form $\rho/3$. Then the numbers (7) are all integers if and only if we have $C \equiv D$, $A \equiv \pm B \pmod 3$. Writing $C = \pm D + 3d$, $A = \mp B + 3b$ in (7), we obtain

$$(9)\qquad \begin{aligned} x &= \rho(3bd \pm bD \mp dB - 2BD), \qquad y = \rho(bD + dB), \\ z &= \rho(2D^2 \pm 2Dd + 3d^2), \qquad w = \rho(2B^2 \mp 2Bb + 3b^2). \end{aligned}$$

For the lower signs, we replace D by l, d by $-q$, B by $-n$, b by r, and obtain (8). For the upper signs, we replace D by $l+q$, d by $-q$, B by $r-n$, b by r, and again obtain (8).

In our next paper we shall show how to construct a machine which examines in this manner each of the infinitude of cases corresponding to the values of the denominators of all irreducible fractions σ, and we shall prove that the solutions which result from any denominator are identical with the solutions (6) and (8) which resulted from the denominators 1 and 2. It will then follow that (6) and (8) together give all the integral solutions of (1) when all the parameters take only integral values.

The goal just reached for our example (1) indicates the desirable form for the integral solutions of any homogeneous equation, viz. expressibility by one or more sets of formulas involving only integral parameters. As in our example, the two sets of formulas (6) and (8) which together give all the integral solutions, may be combined, by way of abbreviation, into a single set of formulas (6) in which the denominator of the only non-integral parameter ρ is limited to the values 1 and 2. Conversely, when it is claimed that all integral solutions of a homogeneous equation† are given by formulas

* Not all solutions (8) are included among solutions (6). When $l = q = n = r = \rho = 1$, (8) gives $x = -1$, $y = 2$, $z = 7$, $w = 3$. If this solution were of the form (6) for integral a, b, c, d, ρ, then $\rho = \pm 1$ by $x = -1$, and, by $z = 7$, $\pm(c^2 + 5d^2) = 7$, which is impossible in integers.

† As to its rational solutions, if we except the rare cases in which recurring series are used, when a homogeneous equation has been completely solved in rational numbers, the unknowns $x_1, x_2, \cdots$ are expressed as homogeneous polynomials f_i, with integral coefficients, of the same degree in certain independent rational parameters $A, B, \cdots$. Writing $A = ka$,

involving a rational parameter, it should be in the sense of an abbreviated statement with explicit indication of easily performable operations leading to formulas containing only integral parameters. Strictly speaking, we do not produce a solution in integers except by a finite number of additions and multiplications performed upon independent integral parameters. These statements are in accord with the evident intention of writers on this subject, even though their conclusions are not proved and very frequently are erroneous.

§ 5. *A Theorem concerning Pairs of Equations.* By way of contrast with § 2, § 3, we note that it is true that the problem of solving any Diophantine equation in rational numbers is equivalent to the problem of solving the corresponding homogeneous equation in rational numbers. In fact, by definition we can pass from the one equation to the other by a substitution like (3). Thus, if x, y, z, w $(w \neq 0)$ give a rational solution of the homogeneous equation, then X, Y, Z give a rational solution of the corresponding equation. Conversely, any rational solution X, Y, Z of the latter gives the solution $x = wX, y = wY, z = wZ, w$ of the homogeneous equation, where w is any rational number. Here there is no delicate question of sorting out solutions of a desired type from those initially obtained.

THE UNIVERSITY OF CHICAGO,
February 15, 1920.

$B = kb, \cdots$, where $a, b, \cdots$ are integers without a common factor, we obtain

$$x_1 = \rho f_1(a, b, \cdots), \qquad x_2 = \rho f_2(a, b, \cdots), \quad \cdots,$$

where ρ alone takes rational values. If a homogeneous equation in three unknowns represents a unicursal curve (of genus zero), its rational solutions must be of this form, as shown by Hilbert and Hurwitz, ACTA MATHEMATICA, vol. 14 (1890–1), pp. 217–24. Some writers have expressed their solutions as non-homogeneous polynomials in parameters $p_1, \cdots, p_k$; to pass to homogeneous polynomials, we have only to use new parameters $P, P_1 = p_1/P, \cdots, P_k = p_k/P$. One writer used parameters subject to an equation of condition, which a later writer solved, and passed to independent parameters. The case of dependent parameters is a preliminary stage in the treatment of the problem. For details on these points, with references, see the writer's *History of the Theory of Numbers*, vol. 2 (*Diophantine Analysis*), Carnegie Institution of Washington, 1920, pp. 556–8, 646, 675–6, 695.

ON THE FOURIER COEFFICIENTS OF A CONTINUOUS FUNCTION.

BY DR. T. H. GRONWALL.

(Read before the American Mathematical Society September 8, 1920.)

It is well known that when

$$\frac{a_0}{2}+\sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta)$$

is the Fourier expansion of a function $f(\theta)$ which is real and continuous for $0 \leqq \theta \leqq 2\pi$, then $\Sigma(a_n^2 + b_n^2)$ converges. Here the exponent 2 cannot in general be replaced by a smaller one; in fact, Carleman* has constructed an example of a continuous $f(\theta)$ where $\Sigma(a_n^{2-2\delta} + b_n^{2-2\delta})$ diverges for any $\delta > 0$, and this example has been simplified by Landau.†

In the present note it will be shown that, *given any single-valued real function $\varphi(x)$, subject only to the condition that $\varphi(x)$ becomes infinite as x becomes infinite, there exists a real continuous function $f(\theta)$ whose Fourier coefficients a_n, b_n make the series*

$$\sum (a_n^2 + b_n^2)\varphi\left(\frac{1}{a_n^2 + b_n^2}\right)$$

divergent. Assuming $\varphi(x) = x^\delta$, where $\delta > 0$, and observing that $(a^2 + b^2)^{1-\delta} < a^{2-2\delta} + b^{2-2\delta}$, we have the particular result referred to above.

If we denote by $f_1(\theta)$ the function conjugate to $f(\theta)$, and write $z = e^{\theta i}$, $F(z) = f(\theta) + if_1(\theta)$, the Fourier expansion of $F(z)$ is $\Sigma_0^\infty c_n z^n$, where $c_0 = a_0/2$, $c_n = a_n - ib_n$ $(n > 0)$. Our statement will be proved by constructing a function $F(z)$ continuous for $|z| = 1$ and such that $\Sigma |c_n|^2 \varphi(1/|c_n|^2)$ diverges. This will be done by means of the following result due to Hardy and Littlewood‡ and used by Landau, loc. cit., for a different purpose:

---

* T. Carleman, *Ueber die Fourierkoeffizienten einer stetigen Funktion*, ACTA MATH., vol. 41 (1918), pp. 377–384.

† E. Landau, *Bemerkungen zu einer Arbeit des Herrn Carleman*, MATHEMATISCHE ZEITSCHRIFT, vol. 5 (1919), pp. 147–153.

‡ G. H. Hardy and J. E. Littlewood, *Some problems of diophantine approximation*, ACTA MATH., vol. 37 (1914), pp. 155–239. See p. 220.

Let ξ be a real irrational number such that all the denominators in its expansion in a continued fraction are bounded (for instance $\xi = \sqrt{2}$ or any quadratic irrationality). Then there exists an $A = A(\xi)$ independent of n and z such that for any $n \geqq 1$, and any z on the unit circle $|z| = 1$,

$$\left| \sum_{\nu=1}^{n} e^{\nu^2 \pi \xi i} z^\nu \right| < A\sqrt{n}.$$

Making

$$F_\nu(z) = \sum_{\mu=1}^{n_\nu} \frac{e^{\mu^2 \pi \xi i}}{\sqrt{n_\nu}} z^\mu,$$

we have therefore $|F_\nu(z)| < A$ for $|z| = 1$; writing $k_\nu = n_0 + n_1 + \cdots + n_{\nu-1}$ and assuming d_ν to be such that $\Sigma|d_\nu|$ converges, we find that the series

$$F(z) = \sum_{\nu=0}^{\infty} d_\nu z^{k_\nu} F_\nu(z)$$

converges uniformly for $|z| = 1$, so that $F(z)$ is continuous on the unit circle. Multiplying by $z^{-n-1}\,dz$ and integrating along the unit circle, we may integrate term by term to the right on account of the uniform convergence, and the Fourier coefficients c_n of $F(z)$ are thus found to be

$$c_n = d_\nu \frac{e^{\mu^2 \pi \xi i}}{\sqrt{n_\nu}} \qquad (n = k_\nu + 1,\ k_\nu + 2,\ \cdots,\ k_\nu + n_\nu).$$

Consequently

$$\sum_{n=k_\nu+1}^{k_{\nu+1}} |c_n|^2 \varphi\left(\frac{1}{|c_n|^2}\right) = |d_\nu|^2 \varphi\left(\frac{n_\nu}{|d_\nu|^2}\right),$$

and since $\varphi(x)$ becomes infinite as x becomes infinite, we may choose each n_ν so that

$$\varphi\left(\frac{n_\nu}{|d_\nu|^2}\right) > \frac{|D_\nu|}{|d_\nu|^2},$$

where $\Sigma|D_\nu|$ is any given divergent series. With this choice of n_ν, it follows that $\Sigma|c_n|^2\varphi(1/|c_n|^2)$ diverges, which proves our theorem.

TECHNICAL STAFF,
OFFICE OF THE CHIEF OF ORDNANCE.

EXTENSION OF AN EXISTENCE THEOREM FOR A NON-SELF-ADJOINT LINEAR SYSTEM.

BY PROFESSOR H. J. ETTLINGER.

(Read before the American Mathematical Society December 31, 1919.)

In a recent paper* the writer established the existence of at least one real characteristic number for the non-self-adjoint system

$$(1) \qquad \frac{d}{dx}\left[K(x,\lambda)\frac{du}{dx}\right] - G(x,\lambda)u = 0,$$

$$(2) \quad U_i = A_{i1}u(a) - A_{i2}K(a)u_x(a) - A_{i3}u(b) + A_{i4}K(b)u_x(b) = 0 \quad (i = 1, 2),$$

satisfying the following conditions I, II, III, IV, V, and either VI A or VI B:

I. $K(x,\lambda)$ and $G(x,\lambda)$ are continuous, real functions of x in $a \leqq x \leqq b$ and for all real values of λ in the interval

$$\Lambda(\Lambda_1 < \lambda < \Lambda_2).$$

II. $K(x,\lambda)$ is positive everywhere in (a, b), Λ.

III. The sets of real constants A_{1j} and A_{2j} are not proportional.

IV. For each value of x in (a, b), K and G do not increase as λ increases.

V. $\lim_{\lambda=\Lambda_1} -\frac{\min G}{\min K} = -\infty, \quad \lim_{\lambda=\Lambda_2} -\frac{\max G}{\max K} = +\infty.$

VI A.† $D_{12}\cdot D_{34} = 0$, together with either

$$(a) \qquad D_{24}^2 + D_{14}^2 \neq 0, \quad D_{14} \geqq 0, \quad D_{24} \geqq 0,$$

or

$$(b) \quad D_{24}^2 + D_{14}^2 = 0, \quad D_{13}^2 + D_{23}^2 \neq 0, \quad D_{13} \leqq 0, \quad D_{23} \leqq 0.$$

VI B. $D_{12}\cdot D_{34} \neq 0$, together with either

$$(a)\ D_{24} > 0 \quad \text{or} \quad (b)\ D_{24} = 0, \quad D_{14} > 0.$$

* *Existence theorem for the non-self-adjoint linear system of the second order*, ANNALS OF MATHEMATICS, vol. 25 (1920), pp. 278–290.

† $D_{ij} = \begin{vmatrix} A_{1i} & A_{1j} \\ A_{2i} & A_{2j} \end{vmatrix}$.

It is the purpose of this note to extend this theorem to the non-self-adjoint system

$$(1) \qquad \frac{d}{dx}\left[K(x,\lambda)\frac{du}{dx}\right] - G(x,\lambda)u = 0,$$

$$(3) \quad U_i = A_{i1}(\lambda)u(a) - A_{i2}(\lambda)K(a)u_x(a) - A_{i3}(\lambda)u(b) + A_{i4}(\lambda)K(b)u_x(b) = 0 \quad (i = 1, 2),$$

satisfying conditions I, II, IV, V, and the following new conditions: III′, either VI′ A or VI′ B, either VII A or VII B or VII C or VII D or VII E, and VIII.

III′. $A_{1j}(\lambda)$ and $A_{2j}(\lambda)$ are two independent sets of functions continuous in λ throughout Λ.

VI′ A. $D_{12}(\lambda) \equiv 0$, $D_{34}(\lambda) \not\equiv 0$, or $D_{12}(\lambda) \not\equiv 0$, $D_{34}(\lambda) \equiv 0$ together with either

(*a*) $D_{24}^2(\lambda) + D_{14}^2(\lambda) \not\equiv 0$, $D_{14}(\Lambda_1 + \epsilon) \geqq 0$, $D_{24}(\Lambda_1 + \epsilon) \geqq 0$,

or

(*b*) $D_{24}^2(\lambda) + D_{14}^2(\lambda) \equiv 0$, $D_{13}^2(\lambda) + D_{23}^2(\lambda) \not\equiv 0$,

$$D_{13}(\Lambda_1 + \epsilon) \leqq 0, \qquad D_{23}(\Lambda_1 + \epsilon) \leqq 0.$$

VI′ B. $D_{12}(\lambda) \cdot D_{34}(\lambda) \not\equiv 0$, together with either

(*a*) $$D_{24}(\Lambda_1 + \epsilon) > 0,$$

or

(*b*) $$D_{24}(\lambda) \equiv 0, \quad D_{14}(\Lambda_1 + \epsilon) > 0.$$

VII *A*.

$$D_{24}(\lambda) \not\equiv 0, \quad \Delta\begin{bmatrix} D_{14}(\lambda) \\ D_{24}(\lambda) \end{bmatrix}^* \leqq 0, \quad \Delta\begin{bmatrix} D_{23}(\lambda) \\ D_{24}(\lambda) \end{bmatrix} \leqq 0.$$

VII *B*.

$$D_{14}(\lambda) \not\equiv 0, \qquad \Delta\begin{bmatrix} D_{14}(\lambda) \\ D_{24}(\lambda) \end{bmatrix} \leqq 0,$$

together with either $D_{24}(\lambda) \equiv 0$, or

$$D_{24}(\lambda) \not\equiv 0 \quad \text{and} \quad \Delta\begin{bmatrix} D_{14}(\lambda) \\ D_{24}(\lambda) \end{bmatrix} \leqq 0.$$

VII *C*.

$$D_{23}(\lambda) \not\equiv 0, \qquad \Delta\begin{bmatrix} D_{13}(\lambda) \\ D_{23}(\lambda) \end{bmatrix} \leqq 0,$$

* $\Delta\phi(\lambda) = \phi(\lambda + \Delta\lambda) - \phi(\lambda)$ for $\Delta\lambda > 0$.

together with either $D_{14}(\lambda) \equiv 0$, or

$$D_{14}(\lambda) \not\equiv 0 \quad \text{and} \quad \Delta\left[\frac{D_{13}(\lambda)}{D_{14}(\lambda)}\right] \leqq 0.$$

VII *D*.

$$D_{23}(\lambda) \not\equiv 0 \quad \text{and} \quad \Delta\left[\frac{D_{13}(\lambda)}{D_{23}(\lambda)}\right] \leqq 0,$$

together with either $D_{24}(\lambda) \equiv 0$, or

$$D_{24}(\lambda) \not\equiv 0 \quad \text{and} \quad \Delta\left[\frac{D_{23}(\lambda)}{D_{24}(\lambda)}\right] \leqq 0.$$

VII *E*.* Either $D_{23}(\lambda) \equiv 0$, or

$$D_{23}(\lambda) \not\equiv 0 \quad \text{and} \quad \Delta\left[\frac{D_{13}(\lambda)}{D_{23}(\lambda)}\right] \leqq 0,$$

together with either $D_{24}(\lambda) \equiv 0$, or

$$D_{24}(\lambda) \not\equiv 0 \text{ and } \Delta\left[\frac{D_{14}(\lambda)}{D_{24}(\lambda)}\right] \leqq 0.$$

VIII. $\Delta D_{13} \cdot \Delta D_{24} - \Delta D_{23} \cdot \Delta D_{14} \leqq 0$.

The types of (3) which we shall consider are the following:

Type I. (*a*) $D_{12}(\lambda) \equiv 0, \quad D_{34}(\lambda) \not\equiv 0,$

(*b*) $D_{12}(\lambda) \not\equiv 0, \quad D_{34}(\lambda) \equiv 0.$

Type II. $D_{12}(\lambda) \cdot D_{34}(\lambda) > 0.$

Type III. $D_{12}(\lambda) \cdot D_{34}(\lambda) < 0.$

Then the various sets of non-self-adjoint conditions† are

$$A. \begin{pmatrix} D_{12}(\lambda) & 0 & D_{32}(\lambda) & D_{42}(\lambda) \\ D_{14}(\lambda) & D_{24}(\lambda) & 0 & 0 \end{pmatrix}$$

$$B. \begin{pmatrix} 0 & D_{12}(\lambda) & D_{13}(\lambda) & D_{24}(\lambda) \\ D_{14}(\lambda) & D_{24}(\lambda) & 0 & 0 \end{pmatrix}$$

$$C. \begin{pmatrix} 0 & D_{12}(\lambda) & D_{13}(\lambda) & D_{14}(\lambda) \\ D_{13}(\lambda) & D_{23}(\lambda) & 0 & 0 \end{pmatrix}$$

$$D. \begin{pmatrix} D_{12}(\lambda) & 0 & D_{32}(\lambda) & D_{42}(\lambda) \\ D_{13}(\lambda) & D_{23}(\lambda) & 0 & 0 \end{pmatrix}$$

$$E. \begin{pmatrix} D_{13}(\lambda) & D_{23}(\lambda) & 0 & D_{43}(\lambda) \\ D_{14}(\lambda) & D_{24}(\lambda) & D_{34}(\lambda) & 0 \end{pmatrix}$$

* Evidently the alternative conditions VII hold for the corresponding set of conditions enumerated later.

† See ANNALS OF MATHEMATICS, loc. cit.

These conditions ensure the validity of the theorems* used in the proof of the existence theorem stated above, and the proof follows exactly as in the original theorem.

UNIVERSITY OF TEXAS,
July 30, 1920.

ON THE CAUCHY-GOURSAT THEOREM.

BY R. L. BORGER.

(Read before the American Mathematical Society December 30, 1919.)

In order to prove his integral theorem, viz: $\int_C f(z)dz = 0$, Cauchy found it necessary to assume not only that the derivative $f'(z)$ existed but also that it was continuous. Later, proofs were given by Goursat and by Moore† in which the mere existence of $f'(z)$ was shown to be sufficient for the truth of the theorem. These were based upon the analysis of the complex variable.

From the standpoint of the real variable many interesting investigations have developed around the Cauchy-Goursat theorem. They have depended upon Green's theorem. Porter,‡ using the Riemann integral, proved that with proper restrictions upon the component functions, U and V, of the complex function, Green's theorem was true, and hence that Cauchy's integral theorem was also true, even when the derivative $f'(z)$ did not exist. Montel,§ by means of the Lebesgue integral, proved Green's theorem under the hypothesis that U_x, V_y, exist, are bounded, and satisfy the equation

$$U_x = V_y,$$

except at most in a set of measure zero. He was then able to prove the integral theorem, and the existence of the derivative $f'(z)$ for a function of the complex variable $f(z)$ in the closed region considered. The existence, then, of the deriva-

* Cf. *Existence theorems for the general, real, self-adjoint linear system of the second order*, TRANSACTIONS AMER. MATH. SOC., vol. 19 (1918), p. 94.

† TRANSACTIONS AMER. MATH. SOCIETY, vol. 1 (1900).

‡ ANNALS OF MATHEMATICS, (2), vol. 7 (1905–6).

§ ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE, (3), vol. 24 (1907).

tive is not necessary for the proof of the integral theorem under the restrictions that were imposed upon the partial derivatives by Montel or by Porter.

It is the purpose of this paper to show that Montel's restrictions as to the boundedness of the partial derivatives are not necessary. By means of the Denjoy integral* it will be shown that the line integral of the ordinary Green's theorem vanishes when taken along all closed curves that lie entirely inside the region and that possess a length. From this it follows that it is possible to establish the Cauchy integral theorem without assuming the existence of $f'(z)$.

We shall prove first the following theorem.

THEOREM. *If $U(x, y)$ and $V(x, y)$ are continuous functions of (x, y) which have finite partial derivatives, U_x, V_y, that satisfy the equation*

$$U_x = V_y$$

in each point of a closed region, R, then

$$(Dn)\int_b^{b'} (Dn)\int_a^{a'} U_x(x, y)dxdy,$$

$$(Dn)\int_a^{a'} (Dn)\int_b^{b'} U_x(x, y)dydx$$

exist and are equal for any rectangular region $a \leqq x \leqq a'$, $b \leqq y \leqq b'$ within R.

Since $U_x(x, y)$ is finite in R, we have

$$(1) \qquad (Dn)\int_a^{x} U_x(x, y)dx = U(x, y) - U(a, y).$$

Then

$$(Dn)\int_b^{y} (Dn)\int_a^{x} U_x(x, y)dxdy$$

$$= (Dn)\int_b^{y} [U(x, y) - U(a, y)]dy$$

exists, since $U(x, y)$ is continuous in (x, y). Similarly, since by hypothesis $U_x = V_y$, we can establish the existence of the integral

* The Denjoy integral will be denoted by the symbol $(Dn)f$. Concerning its properties, see Denjoy, COMPTES RENDUS, vol. 154 (1912), pp. 859–862; and Hildebrandt, this BULLETIN, vol. 24 (1917), pp. 140–144.

$$(Dn)\int_a^{a'}(Dn)\int_b^{b'}U_x(x,y)dydx.$$

To prove the equality of these two double integrals we prove that

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial y}=\frac{\partial}{\partial y}\frac{\partial F}{\partial x}, \tag{2}$$

where

$$F(x,y)=(Dn)\int_b^y(Dn)\int_a^x U_x(x,y)dxdy.$$

Since

$$F(x,y)=(Dn)\int_b^y[U(x,y)-U(a,y)]dy,$$

we have

$$\frac{\partial F}{\partial y}=U(x,y)-U(a,y),$$

whence

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial y}=U_x(x,y)=V_y(x,y). \tag{3}$$

Differentiating $F(x,y)$ first with respect to x, we find

$$\frac{\partial F}{\partial x}=\operatorname*{Lim}_{\Delta x=0}\frac{\Delta F}{\Delta x}=\operatorname*{Lim}_{\Delta x=0}(Dn)\int_b^y\frac{U(x+\Delta x,y)-U(x,y)}{\Delta x}dy$$

$$=\operatorname*{Lim}_{\Delta x=0}(Dn)\int_b^y U_x(x+\theta\Delta x,y)dy.$$

Since $U_x=V_y$, we have

$$\frac{\partial F}{\partial x}=\operatorname*{Lim}_{\Delta x=0}(Dn)\int_b^y U_x(x+\theta\Delta x,y)dy$$

$$=\operatorname*{Lim}_{\Delta x=0}(Dn)\int_b^y V_y(x+\theta\Delta x,y)dy$$

$$=\operatorname*{Lim}_{\Delta x=0}[V(x+\theta\Delta x,y)-V(x+\theta\Delta x,b)]$$

$$=V(x,y)-V(x,b).$$

It follows that

$$\frac{\partial}{\partial y}\frac{\partial F}{\partial x}=V_y(x,y)=U_x(x,y),$$

whence, by (3),

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\frac{\partial F}{\partial x}. \tag{4}$$

Integrating the left member of (4) with respect to x, we have

$$(Dn)\int_a^{a'} \frac{\partial}{\partial x}\frac{\partial F}{\partial y}\,dx = \frac{\partial F}{\partial y}(a', y) - \frac{\partial F}{\partial y}(a, y) = (Dn)\int_a^{a'} U_x(x, y)dx,$$

whence

$$(Dn)\int_b^{b'} (Dn)\int_a^{a'} \frac{\partial}{\partial x}\frac{\partial F}{\partial y} = (Dn)\int_b^{b'} \left[\frac{\partial F}{\partial y}(a', y) - \frac{\partial F}{\partial y}(a, y)\right] dy \tag{5}$$

$$= (Dn)\int_b^{b'} (Dn)\int_a^{a'} U_x(x, y)dxdy$$

$$= F(a', b') - F(a', b) - F(a, b') + F(a, b).$$

Integrating the right member of (4) with respect to y and then with respect to x, we get

$$(Dn)\int_a^{a'} (Dn)\int_b^{b'} \frac{\partial}{\partial y}\frac{\partial F}{\partial x}\,dydx = F(a', b') - F(a', b) - F(a, b') + F(a, b),$$

whence

$$(Dn)\int_b^{b'} (Dn)\int_a^{a'} \frac{\partial}{\partial x}\frac{\partial F}{\partial y}\,dxdy = (Dn)\int_a^{a'} (Dn)\int_b^{b'} \frac{\partial}{\partial y}\frac{\partial F}{\partial x}\,dydx$$

or

$$(Dn)\int_b^{b'} (Dn)\int_a^{a'} U_x(x, y)dxdy = (Dn)\int_a^{a'} (Dn)\int_b^{b'} U_x(x, y)dydx. \tag{6}$$

To prove $\int_C Udy + Vdx = 0$, where C is a closed curve that has a length, we consider first the case in which C is a rectangle. The application of this theorem to the line integral over the rectangle follows from (6) and the equality $U_x = V_y$. We have

$$\int_C Udy + Vdx = 0,$$

where the Denjoy integral reduces to the Riemann integral, since U and V are continuous.

By well known methods* this can be extended to any rectifiable curve bounding a simply connected region. Hence we have the following theorem.

THEOREM. *If U and V are continuous functions of (x, y) possessing finite partial derivatives U_x, V_y, and if*

$$U_x = V_y$$

in a region R, then

$$\int_C U dy + V dx = 0$$

over any rectifiable curve C bounding a simply connected domain lying wholly within R.

The proof of the Cauchy integral theorem is immediate. Moreover if this theorem is satisfied by a continuous function $f(z)$, we know that $f(z)$ is analytic. We may then state our results as follows.

THEOREM. *The necessary and sufficient conditions that $\int f(z)dz = 0$ over any rectifiable curve C bounding a simply connected region lying entirely within a domain R are that the component functions U, V of $f(z) = U + iV$ possess finite partial derivatives in R and that they satisfy the Cauchy-Riemann equations*

$$U_x = V_y, \quad U_y = -V_x.$$

It may be noted that the existence† of $f'(z)$ in any point z_0 requires:

(*a*) The existence of U_x, U_y, V_x, V_y;

(*b*) $U_x = V_y$; $U_y = -V_x$;

(*c*) $\Delta U - \left(\frac{\partial U}{\partial x}\Delta x + \frac{\partial U}{\partial y}\Delta\right)$, $\Delta V - \left(\frac{\partial V}{\partial x}\Delta x + \frac{\partial V}{\partial y}\Delta y\right)$

infinitely small with respect to $|\Delta x| + |\Delta y|$. Hence the hypotheses of the preceding theorem are less restrictive than those of Goursat.‡

OHIO UNIVERSITY,
ATHENS, OHIO.
July 10, 1920.

* Goursat, *Mathematical Analysis*, vol. II, part 1, p. 67.
† Fréchet, NOUVELLES ANNALES, vol. 19 (1919), p. 219.
‡ TRANSACTIONS, loc. cit.

ON A GENERAL ARITHMETIC FORMULA OF LIOUVILLE.

BY PROFESSOR E. T. BELL.

(Read before the American Mathematical Society April 9, 1921.)

1. *Introduction.* In his *History of the Theory of Numbers* (vol. 2, chap. 11), Dickson gives a resumé of the present state of the celebrated general formulas of Liouville in the theory of numbers, and remarks that the only formula for which no proof has been published is (Q) of the sixth article. Some years ago I developed an analytic method of complete generality suitable for dealing with all such arithmetic questions, and I applied it incidentally to the proofs of Liouville's formulas. By means of this method we may immediately paraphrase any identity between elliptic, abelian or theta functions, provided only that it contains the arguments of the functions and is not merely an identity between constants, into another identity between arithmetic functions of the greatest generality, such, for example, as those considered by Liouville.

A detailed account of the general principles upon which the method is given in my paper in the January number of the TRANSACTIONS. In a continuation of the same paper which is to appear later, there is a selection of illustrative examples including some of Liouville's formulas and others of new kinds. Since the formula (Q) is not among these, however, I shall give a proof of it here. This formula is in fact unique among all of Liouville's, for it is the only one which depends immediately upon the addition theorems for the elliptic functions. It will be necessary first to recall a very special case of a general theorem established in the paper just mentioned.

2. *A general Theorem.* Let $f(x, y)$ denote a function of the arguments x, y which takes a single definite value whenever x and y are positive, zero or negative integers, and let the a, b, c, and d denote integers. Then if the following equation is an identity in u and v, $\Sigma_i a_i \sin b_i u \sin c_i v = 0$, we can infer that $\Sigma_i a_i f(b_i, c_i) = 0$, for $f(x, y)$ defined as above and subjected to the conditions

$$f(x, y) = -f(-x, y) = -f(x, -y), \quad f(0, y) = 0 = f(x, 0).$$

Beyond these conditions $f(x, y)$ is general in the widest sense. A proof of this result will be found in the paper cited above.

3. *The Formula (Q) of Liouville.* Let us write for brevity $\mu = 2K/\pi$, $k\mu = h$, and consider the following obvious identity, in which accents signify derivatives with respect to u or with respect to v,

$$\begin{aligned}h\mathrm{sn}\mu(u+v)&[(h\mathrm{sn}\mu u)(h\mathrm{sn}\mu v)'-(h\mathrm{sn}\mu v)(h\mathrm{sn}\mu u)']\\&=h\mathrm{sn}\mu(u-v)[(h\mathrm{sn}\mu u)(h\mathrm{sn}\mu v)'+(h\mathrm{sn}\mu v)(h\mathrm{sn}\mu u)'].\end{aligned}$$

From the classical series for $\mathrm{sn}\mu u$ we have at once

$$h\mathrm{sn}\mu u = 4\Sigma q^{m/2}(\Sigma \sin du), \qquad (h\mathrm{sn}\mu u)' = 4\Sigma q^{m/2}(\Sigma\, d\cos du),$$

where the first summation extends to all $m = 1, 3, 5, \cdots$, and the second to all the divisors $1, d, \cdots, m$ of m. Replacing in the identity the sn, sn' functions by their equivalent series and then equating coefficients of like powers of q, we find after some easy reductions the following identity in u, v,

$$\begin{aligned}\Sigma d_3[&\sin(d_1+d_3)u\sin(d_1+d_2)v-\sin(d_1+d_2)u\sin(d_1+d_3)v\\&+\sin(s_1-d_3)u\sin(d_1+d_2)v-\sin(d_1+d_2)u\sin(d_1-d_3)v]\\=\Sigma d_3[&\sin(d_1+d_3)u\sin(d_1-d_2)v-\sin(d_1-d_2)u\sin(d_1+d_3)v\\&+\sin(d_1-d_3)u\sin(d_1-d_2)v-\sin(d_1-d_2)u\sin(d_1-d_3)v],\end{aligned}$$

where the summations refer to all positive divisors d_1, d_2, d_3 defined by

$$m = m_1 + m_2 + m_3, \qquad m_1 = d_1\delta_1, \qquad m_2 = d_2\delta_2, \qquad m_3 = d_3\delta_3,$$

m, m_1, m_2, m_3 being odd and positive and m constant. Hence by the theorem quoted in § 2,

$$\begin{aligned}\Sigma d_3[&f(d_1+d_3, d_1+d_2)-f(d_1+d_2, d_1+d_3)\\&+f(d_1-d_3, d_1+d_2)-f(d_1+d_2, d_1-d_3)]\\=\Sigma d_3[&f(d_1+d_3, d_1-d_2)-f(d_1-d_2, d_1+d_3)\\&+f(d_1-d_3, d_1-d_2)-f(d_1-d_2, d_1-d_3)].\end{aligned}$$

Now putting $\psi(x, y) = f(x, y) - f(y, x)$, we see that $\psi(x, y)$ has a single determinate value whenever x and y are positive, zero or negative integers, and that $\psi(x, y)$ satisfies the conditions

$$\psi(-x, y) = -\psi(x, y) = \psi(x, -y) = \psi(y, x),$$

$$\psi(0, y) = 0 = \psi(x, 0).$$

Moreover, if $\psi(x, y)$ is the most general function satisfying all of these conditions, we can put $\psi(x, y) = f(x, y) - f(y, x)$ without loss of generality. A brief discussion of such questions is given in section III of the paper cited. Hence from the identity for $f(x, y)$, we infer at once that

$$\Sigma d_3[\psi(d_1 + d_3, d_1 + d_2) + \psi(d_1 - d_3, d_1 + d_2)]$$
$$= \Sigma d_3[\psi(d_1 + d_3, d_1 - d_2) + \psi(d_1 - d_3, d_1 - d_2)],$$

and this is the formula (Q) of Liouville.

The 39 forms of the addition theorems given by Jacobi in section 18 of the *Fundamenta Nova* imply a multitude of such consequences, many of which are of arithmetic interest.

The author wishes to express his indebtedness to Professor Frank Nelson Cole for encouragement and inspiration, not only in this paper, but for much of his other mathematical work.

UNIVERSITY OF WASHINGTON,
January 19, 1921.

SHORTER NOTICES

General Theory of Polyconic Projections. By Oscar S. Adams. Washington, United States Coast and Geodetic Survey, 1919. Special Publication No. 57. 174 pp.

There are many ways of representing, or projecting, the surface of the earth, or parts of it, upon a plane. Any system of lines may be chosen to represent the parallels of latitude, and a second system to represent the meridians. The book before us is designed to give a full account of the so-called polyconic projection, that is the projection in which parallels of latitude are represented by arcs of a non-concentric system of circles with collinear centers. The line of centers is usually, but not necessarily, taken for the central, or principal, meridian. The mathematical problem consists in setting up the equations for the meridians under various hypotheses, methods for constructing the meridians, spacing the parallels, determining the magnification, and so forth. These details the author has worked out for various cases, deriving the formulas for the ellipsoid as well as for the sphere.

Stereographic projection is one type of polyconic projection.

The author devotes considerable space to it, taking for the plane of projection the equatorial plane, or the plane of a meridian, or the plane of the horizon of a given place.

Another important case is that in which a given parallel is represented by a circle whose radius is equal to an element of the tangent cone to the earth's surface along the parallel and included between the point of contact and the vertex. Parallels are spaced along the central meridian in proportion to their true distances along this meridian. This is the case usually referred to as polyconic projection. (See the article on *Mathematical Geography* by Col. Sir A. R. Clarke in the ENCYCLOPAEDIA BRITTANICA.) This projection has been used extensively by the U. S. Coast and Geodetic Survey.

The book might be improved considerably by numbering the formulas and by making the important ones stand out more prominently than is done in the text. For example, the equation for a meridian in the general polyconic projection is derived on page 12 without comment as to its basic importance, indeed without stating that it is the equation for a meridian.

Those who are interested in the practical construction of maps will no doubt find the book of great assistance, and to these it must make its appeal.

L. W. DOWLING.

Einleitung in die Mengenlehre. By A. Fraenkel. Berlin, J. Springer, 1919. iv + 155 pages.

The author proposes to give an introduction to the theory of infinite sets which can be understood by anyone who has sufficient interest and patience. No other prerequisites are set down. The author tells that he had experience in this sort of presentation during the recent war, when he lightened many wearisome hours by explaining *Mengenlehre* to his comrades in the field.

Among the chapter headings are: the concept of set; the concepts of equivalence and infinite sets; countable sets; the continuum; the concept of cardinal number; comparability of cardinal numbers; operations on cardinal numbers; ordered sets and types of order; linear point sets; well-ordered sets, well-ordering and its significance; logical paradoxes and the concept of set.

The choice of topics and the extent to which each topic is treated are well determined. Scientific honesty is not sacrificed

for (apparently) easy assimilation by the reader. Of course, the treatment of many of the above topics must be incomplete in a presentation such as is given in this book, but the author points out any incompleteness in each definition or proof, and suggests possible ways of filling the lacunae.

Some of the exposition is prolix, but prolixity is difficult to avoid in an exposition designed for the general reader. Even with this prolixity the book is very readable. Many examples well illustrate the abstract treatment of the various topics. Some of the more detailed and technical material which may be omitted without destroying the continuity of the exposition is printed in small type.

In the last chapter the author becomes dogmatic in some statements concerning the principle of selection (das Auswahlprinzip) of Zermelo; but enough is said to enable one to see that the author's point of view is not the only possible one. The axiomatic setting up of the theory of sets according to Zermelo, the paradoxes which are to be avoided in this way, and the bearing of the problem of well-ordering on these matters, are explained here quite clearly, considering the limitations imposed by a popular exposition of these abstruse subjects. The method of *logicizing*, and more particularly the theory of types of Russell, are not mentioned, although a footnote reference to Russell's books is given.

The book should be very useful for upper collegiate classes in mathematics and for those interested in mathematical philosophy in a general way. It should help to introduce to a wider circle the ideas and methods of a fundamental and interesting branch of mathematics.

G. A. PFEIFFER.

Descriptive Geometry. By Ervin Kenison and Harry Cyrus Bradley. New York, The Macmillan Company, 1917. vii + 287 pp.

This is one of a series of texts on topics in engineering edited by E. R. Hedrick. In their preface the authors state, "This book represents a teaching experience of more than twenty years on the part of both the authors at the Massachusetts Institute of Technology. . . . The point of view · · · is · · · that of the draftsman. Mathematical formulae and analytic computations have been almost entirely suppressed. . . . The method of attack throughout the book

is intended to be that which shall most clearly present the actual conditions in space. . . . The amount of ground covered is that which is considered sufficient to enable the student to begin the study of the technical drawings of any line of engineering or architecture. It is not intended to be a complete treatise on descriptive geometry. Detailed exposition of such branches as shades and shadows, perspective, stereographic projection, axonometry, the solution of spherical triangles, etc., will not be found."

Questions relating to points, straight lines, and planes occupy the first three-fifths of this text; while the latter part, including twenty-two of the fifty-two problems, is devoted to tangent lines and planes, and to curved surfaces and their intersections. The first two problems,—to find the traces of a straight line, and to find its projections from its traces,—are on pages 24 and 25. The third problem,—to find the true length of a straight line,—is found thirty pages further along, after four chapters on simple shadows, the representation of the plane, the profile plane of projection, and secondary planes of projection. After a chapter on simple intersections and developments, there follow twelve problems grouped into three chapters on lines in a plane and parallel lines and planes, on perpendicular lines and planes, and on the intersections of planes and of lines and planes. The fifteen problems involving the revolution and counter-revolution of planes follow a chapter on the intersection of planes and solids.

Of the ten problems on tangent planes there are three each for cones and cylinders, and two each for spheres and double-curved surfaces of revolution. Then follow problems on the intersection of a plane with a cone, a frustum of a cone, a cylinder, a prism or a pyramid, and a double-curved surface of revolution. The book is concluded by problems on the intersection of two cylinders, of a cylinder and a cone, of two cones, of a sphere and a cone, of two surfaces of revolution whose axes intersect, and of any two curved surfaces.

These problems are carefully worked out with figures, analysis, construction, general case, and special cases. They will probably bring to the student an idea of the general methods employed. The one criticism that the reviewer would suggest is the absence of exercises to be worked out by the student. The book gives, however, an interesting presentation and is quite worth careful study.

E. B. Cowley.

NOTES.

The January number (vol. 43, no. 1) of the AMERICAN JOURNAL OF MATHEMATICS contains the following papers: *Multiple binary forms with the closure property*, by A. B. Coble; *Einstein's theory of gravitation: determination of the field by light signals*, by Edward Kasner; *Note on Einstein's equation of an orbit*, by Frank Morley; *A one-to-one representation of geodesics on a surface of negative curvature*, by H. M. Morse; *Conjugate systems with indeterminate axis curves*, by E. P. Lane.

NATURE has published a special number (vol. 106, no. 2677, February 17, 1921) devoted entirely to discussions of various aspects of relativity theory. It contains papers by Professor A. Einstein, Mr. E. Cunningham, Sir F. Dyson, Dr. A. C. D. Crommelin, Dr. C. E. St. John, Professor G. B. Mathews, Mr. J. H. Jeans, Professor H. A. Lorentz, Sir O. Lodge, Professor H. Weyl, Professor A. S. Eddington, Dr. N. Campbell, Miss Dorothy Wrench and Mr. H. Jeffreys, and Professor H. W. Carr.

The National Research Council requested Professor H. L. Rietz, of the University of Iowa, to call together a small group for a preliminary conference to discuss the desirability of a committee of the Council in the field of the applications of mathematics to statistics, and to make appropriate recommendations. This group consisted of Professors J. W. Glover, E. V. Huntington, Raymond Pearl, and W. M. Persons, and the conference was held in Washington, January 21–22.

The gold medal of the Royal Astronomical Society has been awarded to Professor H. N. Russell, for his contributions to the theory of stellar evolution.

The Paris Academy of Sciences has awarded 3000 francs from its Loutreuil Foundation to Henri Brocard and Léon Lemoyne, for the publication of the second and third volumes of their work entitled *Courbes géométriques remarquables planes et gauches*.

At the University of Toulouse, Professor Buhl has at his own request been transferred from the chair of general mathematics to that of the differential and integral calculus.

Canon J. M. Wilson has been elected president of the British Mathematical Association.

At the Massachusetts Institute of Technology, Assistant Professor Joseph Lipka has been granted leave of absence for the year 1921–1922. He intends to study in Rome.

At Cornell University Mr. H. S. Vandiver has been awarded a subvention from the Heckscher Research Foundation to support investigations on the theory of algebraic numbers.

At Westminster College, New Wilmington, Pa., Associate Professor J. V. McKelvey, of Iowa State College, has been appointed Professor.

Professor Georges Humbert, of the Ecole Polytechnique and the Collège de France, member of the Paris Academy of Sciences in the section of geometry, died January 22, 1921, at the age of sixty-two years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMODEO (F.). Lezioni di geometria proiettiva. 3a edizione. Napoli, L. Pierro, 1920.

BJORLING (C. F. E.). Oevningsuppgifter i differentialkalkyl och algebraisk analys. Lund, Gleerup, 1920. 8vo. 63 pp.

BOHR (H.), BONNESEN (T.), MOLLERUP (J.) og PÀL (J.). Elementerne af de algebraiske Tals Teori. Fire Foredrag holdt i Matematisk Forening (København). København, Gjellerup, 1920. 8vo. 115 pp.

BOLZANO (B.). Paradoxien des Unendlichen. Mit Unterstützung der Gesellschaft zur Förderung Deutscher Wissenschaft, Kunst und Literatur in Böhmen herausgegeben von der Philosophischen Gesellschaft an der Universität Wien durch Alois Höfler. Mit Anmerkungen versehen von Hans Hahn. (Philosophische Bibliothek, Band 99.) Leipzig, Meiner, 1921. 12mo. 12 + 157 pp.

BONNESEN (T.). See BOHR (H.).

BREMEKAMP (H.). De practische en de critische richting in de wiskunde. Rede uitgesproken bij de aanvaarding van het ambt van hoogeleeraar aan de Technische Hoogeschool te Delft. Groningen, Noordhoff, 1919. 19 pp.

BURALI-FORTI (C.). Fondamenti per la geometria del triangolo. Palermo, Capozzi, 1919.

CRUEWELL (E. R.). Die beiden Grundprobleme der Mathematik.
A: Das Problem der mathematischen Messkunst. Der Satz von Pythagoras.
B: Das Problem der Differentialrechnung. Die Brücke des Differentials.
Gemeinverständliche Abteilung beider Probleme mittelst einfacher Buchstabenrechnung. Berlin, Selbstverlag, 1920. 8vo. 8 pp.

FRICKE (R.). See KLEIN (F.).

HAHN (H.). See BOLZANO (B.).

HILTON (H.). Plane algebraic curves. Oxford, Clarendon Press, 1920. 16 + 388 pp. 28 s.

HÖFLER (A.). See BOLZANO (B.).

KLEIN (F.). Gesammelte mathematische Abhandlungen. Band 1: Liniengeometrie. Grundlegung der Geometrie zum Erlanger Programm. Herausgegeben von R. Fricke und A. Ostrowski. Berlin, Springer, 1921.

LEFEBVRE (B.). Notes d'histoire des mathématiques. (Antiquité et moyen-âge.) Louvain, Société scientifique de Bruxelles, 1920. 8vo. 154 pp.

MOLLERUP (J.). See BOHR (H.).

OSTROWSKI (A.). See KLEIN (F.).

PÀL (J.). See BOHR (H.).

POSTIGLIONE (S.). La quadratura del circolo. Milano, Tipografia Volontà, 1920.

RIEMANN (B.). Ueber die Hypothesen welche der Geometrie zu Grunde liegen. Neu herausgegeben und erläutert von H. Weyl. 2te Auflage. Berlin, Springer, 1921.

SCHREK (D. J. E.). Beginselen der analytische meetkunde. Groningen, Noordhoff, 1919. 8vo. 144 pp.

TURC (A.). Introduction élémentaire à la géométrie Lobatschewskienne. Ouvrage posthume, publié d'après les notes de l'auteur. Genève, Kündig, 1914. 8vo. 170 pp.

VIVANTI (G.). Lezioni di analisi infinitesimale. 2a edizione. Parte 1a. Torino, Lattes, 1920. 8vo. 7 + 693 pp. L. 50.00

DE VRIES (H). "De vierde dimensies," eene inleiding tot de vergelijkende studie der verschillende meetkunden. Groningen, Noordhoff, 1915. 8vo. 12 + 142 pp.

WEYL (H.). See RIEMANN (B.).

WILLERS (A.). Graphische Integration. (Sammlung Göschen, Nr. 801.) Berlin, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1920.

II. ELEMENTARY MATHEMATICS.

BÜTZBERGER (F.). Lehrbuch der Arithmetik und Algebra für Mittelschulen. 1ter und 2ter Teil. 2te Auflage. Zürich, Art. Institut Orell Füssli, 1920.

FRIIS-PETERSEN (F.) og JESSEN (J. L. W.). Aritmetik og Algebra for det toaarige Præliminærkursus. 8vo. København, Gjellerup, 1919–1920. 63 + 63 pp.

GODFREY (C.) and SIDDONS (A. W.). Notes and answers to exercises in practical geometry and theoretical geometry. Cambridge, University Press, 1920. 26 pp. 1s. 6d.

HARVEY (F. W.). Everyman's mathematics. London, Methuen, 1920. 138 pp. 4s.

HENKLER (P.). Rechnen und Raumlehre in der Volkshochschule. Berlin, Union, 1920.

JACOB (J.). Arithmetik. 1ter Teil, Unterstufe, 4te Auflage. 2ter Teil, Mittelstufe, 2te Auflage. Wien, Deuticke, 1917.

——. Schlüssel, Jacobs Lehrbuch der Arithmetik. Die Theorie in Frageform und die Resultate der Aufgaben. 2ter Teil, für die 4te und 5te Klasse. Wien, Deuticke, 1917.

JESSEN (J. L. W.). See FRIIS-PETERSEN (F.).

LIETZMANN (W.) und ZÜHLKE (P.). Aufgabensammlung und Leitfaden der Geometrie. Ausgabe B, für Realanstalten. Oberstufe. Leipzig, Teubner, 1919.

SIDDONS (A. W.). See GODFREY (C.).

WOLLETZ (K.). Arithmetik und Algebra für die 6te, 7te und 8te Klasse der Gymnasien und Realgymnasien. Wien, Pichlers Witwe und Sohn, 1917. 4 + 221 pp.

ZÜHLKE (P.). See LIETZMANN (W.).

III. APPLIED MATHEMATICS.

ADAMS (O. S.). See BOWIE (W.).

ARIÈS (E.). Thermodynamique. Propriétés générales des fluides. Paris, Hermann, 1920.

BELLASIS (E. S.). Hydraulics with working tables. 3d edition. London, Chapman and Hall, 1920. 8 + 348 pp. 18 s.

BOQUET (F.). Tables du mouvement képlérien. 1ère partie. Paris, Hermann, 1920. 8vo. 6 + 205 pp.

BORN (M.). Der Aufbau der Materie. Drei Aufsätze über moderne Atomistik und Elektronentheorie. Berlin, Springer, 1920.

BOUASSE (H.). Pendule, spiral, diapason. Tome 2. Paris, Delagrave, 1920. 8vo. 22 + 518 pp.

BOWIE (W.) and ADAMS (O. S.). Grid system for progressive maps in the United States. (U. S. Coast and Geodetic Survey, Special Publication No. 59.) Washington, 1919. 227 pp.

BOWLEY (A. L.). Elements of statistics. London, P. S. King, 1921. 11 + 459 pp.

DÉVÉDEC (P.). Application de la résistance des matériaux au calcul des ouvrages de béton armé. Paris, Dunod, 1920. 8vo. 16 + 372 pp. Fr. 40.00

EINSTEIN (A.). Ueber die spezielle und die allgemeine Relativitätstheorie. 6te Auflage. Braunschweig, Vieweg, 1920. 8vo. 4 + 83 pp.

FORREST (S. M.). Mathematics for technical students. Junior course. With answers. London, Arnold, 1920. 8 + 260 pp.

FRISCHAUF (J.). Beiträge zur Landesaufnahme und Kartographie des Erdsphäroids. Leipzig, Teubner, 1919.

GARTELMANN (H.). Zur Relativitätslehre. Berlin, Verlag der "Neuen Weltsanchauung," 1920.

GUILLOT (L.). Cours de mécanique. Tome 1. 2e édition, revue et mise à jour. Paris et Liège, Béranger, 1920. 8vo. 451 pp. Fr. 36.00

INSOLERA (F.). Lezioni di statistica metodologica. Torino, Fratelli Treves, 1921. 8vo. 192 pp. L. 24.00

JACOB (L.). La résistance de l'air et l'expérience. Les conséquences. Paris, Doin, 1921. 2 volumes. 16mo. 586 pp.

JEANS (J. H.). The mathematical theory of electricity and magnetism. 4th edition. Cambridge, University Press, 1921. 24 s.

LINDOW (M.). Differentialrechnung unter Berücksichtigung der praktischen Anwendung in der Technik. 3te Auflage. Leipzig, Teubner, 1919.

MILLER (K.). Die Erdmessung im Altertum und ihr Schicksal. Stuttgart, Strecker und Schröder, 1919.

NEWTON (E. A.). Statistical graphs. New York, American Book Company, 1921.

REICHE (F.). Die Quantentheorie. Ihr Ursprung und ihre Entwicklung. Berlin, Springer, 1921. 6 + 231 pp.

REICHENBACH (H.). Relativitätstheorie und Erkenntnis a priori. Berlin, Springer, 1920.

SCHEFFERS (G.). Lehrbuch der dartsellenden Geometrie. 2ter Band. Berlin, Springer, 1920.

SCHELLENBERG (K.). Anwendung der Integralgleichungen auf die Theorie der Elektrolyse. (Diss., Göttingen.) Leipzig, Barth, 1915. 53 pp.

SCHLICK (M.). Raum und Zeit in der gegenwärtigen Physik. Zur Einführung in das Verständnis der Relativitäts- und Gravitationstheorie. 3te, vermehrte Auflage. Berlin, Springer, 1920.

STUYVAERT (M.). Statique. Dynamique. Ghent, Van Rysselberghe et Rombaut, 1920. 8vo. 205 pp. Fr. 20.00

WEYL (H.). Raum, Zeit, Materie. Vorlesungen über allgemeine Relativitätstheorie. 3te, umgearbeitete Auflage. Berlin, Springer, 1920.

YOUNG (A. E.). Some investigations in the theory of map projections. London, Royal Geographical Society, 1920. 8 + 76 pp.